Spontaneous symmetry breaking and principal fibre bundles

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Abstract. *Under the unique physical existance assumption for an interacting vacuum, a global geometrical construction is given for the Higgs mechanism in the case of spontaneous symmetry breaking in a general compact symmetry group. The mass matrix for the resulting massive gauge bosons in exhibited.*

INTRODUCTION

Since about twenty years, classical gauge theories are well described through differential geometry: a gauge field is generally represented by a connection on a principal fibre bundle P for which the structure group is the symmetry group of the theory.

The formation of massive gauge bosons in the case of spontaneous symmetry breaking is widely discussed in recent literature ([2], [4], [6], [7], [12]) and explained by the «Higgs mechanism»: massive gauge vector bosons would be created by the vanishing, from the physical spectrum, of some (massless) «gauge bosons» and of massless scalar fields (the Goldstone bosons). This picture, unfortunately, breaks the geometrical nature of the theory, since it can only be realized by using particular cross-sections of the principal bundle P (unitary gauges).

In the present paper we discuss geometrical consequences of spontaneous symmetry breaking: to the physical assumption of the existence of vacuum (interacting field of minimal energy) is canonically associated a reduction of the

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principal bundle P. The pull-back on this subbundle of a gauge field on P induces a tensorial 1-form which is interpreted as a system of neutral massive vector particles.

The results are stated in the most general frame, in which the symmetry group is a compact Lie group which can be associated to any non gravitational interaction and where the potential for scalar fields is only assumed to have degenerate minimum. We give a simple and general form to the mass matrix for the massive gauge bosons coming from the spontaneous symmetry breaking.

We remark that the scalar fields potential is used only to select a non trivial orbit of the group of symmetries (hidden or not).

For previous geometric approaches of spontaneous symmetry breaking one case see [10], [5]. The construction developed in [5] is close to ours but made under much stronger assumptions on the symmetry group G and the little group H. These assumptions don't seem to be satisfied in some physical situations, especially in grand-unification theory.

§1. SETTINGS

Let's assume to be given:

(a) a compact, connected Lie group G associated to an interaction called a G-interaction (we do not consider the gravitational interaction).

A positive definite Ad-invariant scalar product $(\cdot|\cdot)$ is chosen, once for all, on the Lie algebra $\mathcal{G}$ of G. If G is semi-simple, one takes usually the opposite of the Killing form.

(b) a unitary representation ρ of G in a hermitian (or euclidean) space $(E, \langle\cdot|\cdot\rangle)$ of dimension n and such that G leaves invariant no non-vanishing vector of E.

(c) a flat space-time V_4 supposed to be an open subset of the Minkowski space $\mathbb{R}^4$ with natural coordinates $(x^\alpha)_{\alpha=0,\dots,3}$ in which components of the flat lorentzian metric are:

$$\eta_{\alpha\beta} = 2\,\delta_\alpha^0\delta_\beta^0 - \delta_{\alpha\beta} \tag{1}$$

(d) a lagrangian density $\mathcal{L}_0 : E \times L(\mathbb{R}^4, E) \to \mathbb{R}$ for n-tuples of scalar fields, written:

$$\mathcal{L}_0(v, w) = \eta^{\alpha\beta}\langle w_\alpha | w_\beta\rangle - V(v) \tag{2}$$

where $w_\alpha = w\left(\dfrac{\partial}{\partial x^\alpha}\right)$ and where $V : E \to \mathbb{R}$ is a potential function assumed to be G invariant. G is called usually the internal symmetry group for the lagrangian density $\mathcal{L}_0$.

The physical assumption of the existence of free fields $\varphi : V_4 \to E$ of minimal energy is equivalent to the statement that the potential V has an absolute minimum which we shall assume to be zero.

(e) a principal fibre bundle $P \xrightarrow{\pi} V_4$ with structure group G and base manifold V_4.

P is called the space of phase factors [14]. An equivariant map $\psi : P \to E$ (i.e. $\psi(pg) = \rho(g^{-1})\, \psi(p)$) represents a n-tuple of scalar fields with G-interaction. An alternative way for description of such fields is to consider them as cross-sections of the associated vector bundle $P \times_\rho E \to V_4$ ([8], [14]).

A G-interaction is defined by a connection form ω on P which is called the gauge field or Yang-Mills field.

The lagrangian density for the pair (ψ, ω) is written:

$$\mathscr{L}_{(\psi,\omega)} = \mathscr{L}_0(\psi, D^\omega \psi) + \mathscr{L}_{YM}(\omega) \tag{3}$$

where:

– for $p \in P$

$$D^\omega \psi(p) = \partial \psi(p) \circ (\partial \pi(p)|_{H_p})^{-1} \tag{4}$$

is the covariant derivative of ψ with respect to the connection ω (H_p is the horizontal space of ω at the point p),

– the Yang-Mills lagrangian density:

$$\mathscr{L}_{YM}(\omega) = -\frac{1}{4}\, \eta^{\alpha\gamma} \eta^{\beta\delta} (F_{\alpha\beta} | F_{\gamma\delta}) \tag{5}$$

is expressed by the means of gauge field strengths [14] which are deduced from the curvature form Ω of the connection ω by:

$$F_{\alpha\beta} = \Omega\left(\frac{\bar\partial}{\partial x^\alpha}, \frac{\bar\partial}{\partial x^\beta}\right) \tag{6}$$

where $\dfrac{\bar\partial}{\partial x^\alpha}$ is the horizontal lift (for ω) of the vector field $\dfrac{\bar\partial}{\partial x^\alpha}$.

Let us notice that the function $\mathscr{L}_{(\psi,\omega)}$ which is a priori defined on P is, according to G-invariance properties, a function on the space-time V_4.

§2. GEOMETRIC CONSEQUENCES OF SPONTANEOUS SYMMETRY BREAKING

The invariance of the potential V implies that the set $V^{-1}(0)$ of points of E where V is minimal is a union of orbits of G.

From now *we assume that* $V^{-1}(0)$ *is a single orbit of* G.

The spontaneous symmetry breaking appears when $k = \dim(V^{-1}(0)) \geqslant 1$.

For all the following, a vacuum state $v_0 \in V^{-1}(0)$ is chosen. Symmetries spontaneously broken are transformations $g \in G$ such that

$$\rho(g)v_0 \neq v_0.$$

The *little group of symmetries* H (defined up to a conjugation in G) is the isotropy subgroup of v_0 in G.

The main assumption (physically reasonable) on which is based the following construction is:

Main assumption - There exists an interacting scalar field $\psi_0 : P \to E$ which takes its values in $V^{-1}(0)$ (interacting vacuum).

The next proposition shows that this assumption has interesting geometric consequences. [8].

PROPOSITION 1. *The set* $P_0 = \psi_0^{-1}(v_0) \subset P$ *is a principal subbundle of* P *with structure group the «little group»* H.

Proof. Let us denote by $\rho' : \mathcal{G} \to L(E)$ the representation of the Lie algebra $\mathcal{G}$ derived from the representation ρ of G.

Let ζ be a tangent vector to $V^{-1}(0)$ at v_0. Then there exists $\xi \in \mathcal{G}$ such that

$$\zeta = \rho'(\xi)v_0.$$

Since ψ_0 is an equivariant map, one sees that for any $p \in P_0$:

$$\zeta = T_p \psi_0 \left(\frac{d}{dt} (p \exp(-t\xi)|_{t=0} \right).$$

Hence, $\psi_0 : P \to V^{-1}(0)$ is a submersion at any point of P_0 and P_0 is a closed submanifold of P such that:

$$\dim P_0 = \dim P - k.$$

On the other hand one can easily verify that

(i) $\forall x \in V_4 : P_0 \cap \pi^{-1}(x) \neq \phi,$

(ii) $\forall p \in P_0, \quad \forall g \in G, \quad pg \in P_0 \text{ iff } g \in H.$ ■

Let us notice that if ω is the extension to P of a 1-form of connection on P_0, one have:

$$D^{\omega} \psi_0 = 0.$$

For an interaction defined by such a connection ω the energy of the scalar field ψ_0 is zero.

Let $\mathscr{K}$ be the orthogonal complement of $\mathscr{H}$ in $\mathscr{G}$ with respect to the Ad--invariant euclidean product $(\cdot|\cdot)$. $\mathscr{K}$ is invariant by the adjoint representation of H in $\mathscr{G}$ and we denote by μ the induced representation of H in $\mathscr{K}$.

Let us give a connection form ω on P defining a G-interaction. Then its pull-back $i^*\omega$ by the canonical imbedding $i = P_0 \hookrightarrow P$ naturally splits into:

$$i^*\omega = \omega_0 + \gamma \tag{7}$$

where ω_0 is $\mathscr{H}$-valued and γ is $\mathscr{K}$-valued. One can easily prove the following proposition:

PROPOSITION 2. *ω_0 is a connection 1-form on P_0 and γ is a tensorial 1-form* ([9], [8]) *of type μ.*

The next proposition will be used in proposition 4 in view of further interpretation of the lagrangian density $\mathscr{L}_{(\psi_0,\omega)}$.

PROPOSITION 3. *Let $p \in P_0$, $Y \in T_{\pi(p)} V_4 = \mathbb{R}^4$. If $\overline{Y} \in T_p P$ is the horizontal lift of Y for ω and $\overline{Y}_0 \in T_p P_0$ is the horizontal lift of Y for ω_0, then:*

$$\overline{Y} = \overline{Y}_0 + \frac{\mathrm{d}}{\mathrm{d}t}\, p \exp(-t\,\gamma(p)(\overline{Y}_0)\,|_{t=0}. \tag{8}$$

Proof. We set:

$$\mathscr{K}_p = \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\, p \exp(t\xi)\,|_{t=0} \,|\, \xi \in \mathscr{K} \right\} \subset T_p P.$$

If

$$X = \frac{\mathrm{d}}{\mathrm{d}t}\, p \exp(t\xi)\,|_{t=0}$$

belongs to $T_p P_0 \cap \mathscr{K}_p$, we have:

$$0 = \partial \psi_0(p)(X) = \frac{\mathrm{d}}{\mathrm{d}t}\, \psi_0(p \exp(t\xi))\,|_{t=p} = -\rho'(\xi) v_0.$$

Hence:

$$\xi \in \mathscr{H} \cap \mathscr{K} = \{0\} \text{ and } X = 0.$$

Since dim $\mathscr{K}_p = k$ we deduce that:

$$T_p P = T_p P_0 \oplus \mathscr{K}_p. \tag{9}$$

Hence:

$$\overline{Y} = X + \frac{\mathrm{d}}{\mathrm{d}t}\, p \exp(t\xi)\big|_{t=0},\ X \in T_p P_0, \quad \xi \in \mathscr{K}. \tag{10}$$

Putting (10) in:

$$\begin{cases} \omega(p)(\overline{Y}) = 0 \\ \partial\, \pi(p)(\overline{Y}) = Y \end{cases}$$

one obtains easily:

$$\begin{cases} X = \overline{Y}_0 \\ \xi = -\gamma(p)(\overline{Y}_0). \end{cases} \tag{11}$$

■

PROPOSITION 4. *Let $p \subset P_0$ and $X, Y \in T_{\pi(p)} V_4$.*
Then:

$$D^{\omega}\psi_0(p)(Y) = \rho'(\gamma(p)(\overline{Y}_0))v_0, \tag{12}$$

$$\Omega(p)(\overline{X}, \overline{Y}) = \Omega_0(p)(\overline{X}_0, \overline{Y}_0) + D^{\omega_0}\gamma(p)(\overline{X}_0, \overline{Y}_0) + [\gamma(p)\overline{X}_0, \gamma(p)\overline{Y}_0]. \tag{13}$$

Where:

(a) Ω (resp. Ω_0) is the curvature 2-form of ω (resp. ω_0);

(b) $\overline{X}$ and $\overline{Y}$ (resp. $\overline{X}_0$ and $\overline{Y}_0$) are the horizontal lifts at p of X and Y for ω (resp. for ω_0);

(c) The tensorial 2-form $D^{\omega_0}\gamma$ is the covariant differential of γ with respect to the connection ω_0.

Proof.

(i) By definition and using proposition 3 we have

$$D^{\omega}\psi_0(p)(Y) = \partial\psi_0(p)(\overline{Y}) = \partial\psi_0(p)\left(\frac{\mathrm{d}}{\mathrm{d}t}\, p \exp(-t\gamma(p)(\overline{Y}_0))\big|_{t=0}\right) =$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\,\rho(\exp(t\gamma(p)(\overline{Y}_0))v_0)\big|_{t=0} = \rho'(\gamma(p)(\overline{Y}_0))v_0.$$

(ii) Since $\Omega = \mathrm{d}\omega + [\omega, \omega]$ and $\overline{X}$ and $\overline{Y}$ are ω-horizontal vectors, we have from proposition 3:

$$\Omega(p)(\bar{X}, \bar{Y}) = \mathrm{d}\omega(p)(\bar{X}, \bar{Y}) = \mathrm{d}\omega(p)(\bar{X}_0, \bar{Y}_0) +$$

$$+ \mathrm{d}\omega(p)\left(\bar{X}, \frac{\mathrm{d}}{\mathrm{d}t} p \exp(-t\gamma(p)(\bar{Y}_0))\,|_{t=0}\right) +$$

$$+ \mathrm{d}\omega(p)\left(\frac{\mathrm{d}}{\mathrm{d}t} p \exp(-t\gamma(p)(\bar{X}_0))\,|_{t=0}, \bar{Y}_0\right).$$

But $\bar{X}_0$ and $\bar{Y}_0$ are tangent to P_0, so the first term is equal to:

$$i^*\mathrm{d}\omega(p)(\bar{X}_0, \bar{Y}_0) = \mathrm{d}\omega_0(\bar{X}_0, \bar{Y}_0) + \mathrm{d}\gamma(p)(\bar{X}_0, \bar{Y}_0) =$$

$$= \Omega_0(p)(\bar{X}_0, \bar{Y}_0) + D^{\omega_0}\gamma(p)(\bar{X}_0, \bar{Y}_0).$$

Since $\bar{X}$ is ω-horizontal, the second term writes:

$$\Omega(p)\left(\bar{X}, \frac{\mathrm{d}}{\mathrm{d}t} p \exp(-t\gamma(p)(\bar{Y}_0))\,|_{t=0}\right)$$

and is equal to 0 because Ω is a tensorial form. In the same way, the third term writes:

$$-\left[\omega\left(\frac{\mathrm{d}}{\mathrm{d}t} p \exp(-t\gamma(p)(\bar{X}_0))\,|_{t=0}\right), \omega(p)(\bar{Y}_0)\right] =$$

$$= [\gamma(p)(\bar{X}_0), i^*\omega(p)(\bar{Y}_0)] = [\gamma(p)(\bar{X}_0), (\gamma(p)(\bar{Y}_0)]. \qquad \blacksquare$$

§4. MASSIVE GAUGE BOSONS

The lagrangian density $\mathscr{L}_{(\psi_0,\,\omega)}$ describing the interacting energy minimizing field ψ_0 and gauge field ω is:

(14) $$\mathscr{L}_{(\psi_0,\,\omega)} = \eta^{\alpha\beta}\langle D^{\omega}_{\alpha}\psi_0 | D^{\omega}_{\beta}\psi_0\rangle + \mathscr{L}_{YM}(\omega)$$

where $\mathscr{L}_{YM}(\omega)$ is given by (5).

From proposition 4, one gets, for $p \in P_0$:

(15) $$D^{\omega}_{\alpha}\psi_0(p) = \rho'(\gamma_{\alpha}(p))v_0,$$

(16) $$F_{\alpha\beta}(p) = F^0_{\alpha\beta}(p) + D^{\omega_0}\gamma_{\alpha\beta}(p) + [\gamma_{\alpha}(p), \gamma_{\beta}(p)]$$

where:

$$\gamma_{\alpha}(p) = \gamma(p)\left(\left(\overline{\frac{\partial}{\partial x^{\alpha}}}\right)_0\right),$$

$$F^0_{\alpha\beta} = \Omega_0(p)\left(\left(\frac{\overline{\partial}}{\partial x^\alpha}\right)_0, \left(\frac{\overline{\partial}}{\partial x^\beta}\right)_0\right),$$

$$D^{\omega_0}\gamma_{\alpha\beta}(p) = D^{\omega_0}\gamma(p)\left(\left(\frac{\overline{\partial}}{\partial x^\alpha}\right)_0, \left(\frac{\overline{\partial}}{\partial x^\beta}\right)_0\right).$$

Putting (15) and (16) in formula (14) we see that the lagrangian density $\mathscr{L}_{(\psi_0,\,\omega)}$ can be expressed as follow

$$\mathscr{L}_{(\psi_0,\,\omega)}(x) = -\frac{1}{4}\,\eta^{\alpha\lambda}\eta^{\beta\delta}(D^{\omega_0}\gamma_{\alpha\beta}(p)\,|\,D^{\omega_0}\gamma_{\lambda\delta}(p)) +$$

(17) $$+\,\eta^{\alpha\beta}\langle\rho'(\gamma_\alpha(p))v_0\,|\,\rho'(\gamma_\beta(p))v_0\rangle + \mathscr{L}_{YM}(\omega_0) +$$

$$+ \text{ terms of degree 3 and 4 in } (\omega_0, \gamma),$$

where p is any point of P_0 over $x \in V_4$.

The first term of (17) shows that the tensorial form γ represents a k-tuple of neutral vector boson fields interacting with the gauge field defined by the connection ω_0 ([1], [11], [15]).

The second term of (17) can be interpreted, as usually done in field theory as a mass term for these vector bosons. In order to have a better understanding of this quadratic expression let us consider an orthonormal basis $(e_1, \ldots, e_k)$ in $\mathscr{K}$ and put:

$$\gamma_\alpha = \gamma^i_\alpha\, e_i.$$

Then, the mass term takes the form:

(18) $$\frac{1}{2}\,\eta^{\alpha\beta}\gamma^i_\alpha(p)\,\gamma^j_\beta(p)(N_{ij})$$

where

(19) $$N_{ij} = 2Re\,\langle\rho'(e_i)v_0\,|\,\rho'(e_j)v_0\rangle.$$

The real symmetric matrix $N = (N_{ij})_{1\leqslant i,j\leqslant k}$ is positive. Indeed, for $(x^1, \ldots, x^k) \in \mathbb{R}^k$ one has:

$$N_{ij}\,x^i\,x^j = 2\,\langle\rho'(x^i e_i)v_0\,|\,\rho'(x^j e_j)v_0\rangle \in \mathbb{R}_+ .$$

If $N_{ij}\,x^i\,x^j = 0$, then:

$\rho'(x^i e_i)v_0 = 0$ so that $x^i e_i \subset \mathscr{H} \cap \mathscr{K} = \{0\}$ and $x^1 = x^2 = \ldots = x^k = 0$.

The real symmetric positive matrix $M = \sqrt{N}$ is the mass matrix for the massive neutral vector bosons represented by γ.

Let us conclude by few ending remarks:

1. Spontaneous symmetry breaking and the existence of an interacting material field potential-minimizing implies that the Yang-Mills field defining the interaction is partly made of massive neutral vector bosons. The latter are called massive gauge bosons and their masses depend only on the representation ρ of the internal symmetry group (see (19)). Moreover, the preceeding study shows that there is no need for introducing the so called Goldstone bosons in order to generate massive gauge bosons.

2. The developments of §2 show that there is a significant geometric difference between the massive gauge bosons and the «remaining massless gauge vector bosons» considered by physicists. In our terminology the first ones are represented by the tensorial 1-form γ while the second ones are deduced from the connection form ω_0 and, by the way, should not, from a geometric point of view, be considered as vector particles. The geometric difference is apparently independant of the physical difference concerning the mass.

3. It is possible to substitute to the Minkowski space-time an arbitrarily given space-time.

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